2.5 Solving Differential Equations

Solution of the general differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Initial conditions:

$$y(t_0) \ , \quad \left.\frac{dy(t)}{dt}\right|_{t=t_0} \ , \quad \ldots, \quad \left.\frac{d^{N-1}y(t)}{dt^{N-1}}\right|_{t=t_0}$$

General solution:

$$y(t) = y_h(t) + y_p(t)$$

- $y_h(t)$ is the *homogeneous solution* of the differential equation (natural response).
- $y_p(t)$ is the *particular solution* of the differential equation.
- $y(t) = y_h(t) + y_p(t)$ is the *forced solution* of the differential equation (forced response).

2.5 Solving Differential Equations

As a mathematical function, $y_h(t)$ is the *homogeneous solution* of the differential equation. From the perspective of the output signal of a system, $y_h(t)$ is called the *natural response* of the system.

$y_h(t)$ depends on the structure of the system as well as the *initial state* of the system. ***It does not depend on the input signal*** applied to the system. It is the part of the response that is produced by the system due to a release of the ***energy stored within the system***.

Recall the circuits used in Examples 2.4 through 2.6. Some ***circuit elements such as capacitors and inductors*** are capable of storing energy which could later be released under certain circumstances.
If at some point in time, a capacitor or an inductor with stored energy is given a chance to release this energy, the circuit could produce a response through this release even when there is no external input signal being applied.

2.5 Solving Differential Equations

In contrast, the second term $y_p(t)$ is part of the solution that is *due to the input signal x(t)* being applied to the system. It is referred to as the ***particular solution*** of the differential equation.

It depends on the input signal x (t) and the internal structure of the system, but ***it does not depend on the initial state of the system***. It is the part of the response that remains active after the homogeneous solution $y_h(t)$ gradually becomes smaller and disappears.

the ***steady-state response*** of the system, that is, the response to an input signal that has been applied for a long enough time for the transient terms to die out.

2.5 Solving Differential Equations

Finding the natural response of a continuous-time system

Homogeneous differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

First-order homogeneous differential equation:

$$\frac{dy(t)}{dt} + \alpha\, y(t) = 0$$

Solution:

$$y(t) = c\, e^{-\alpha t}$$

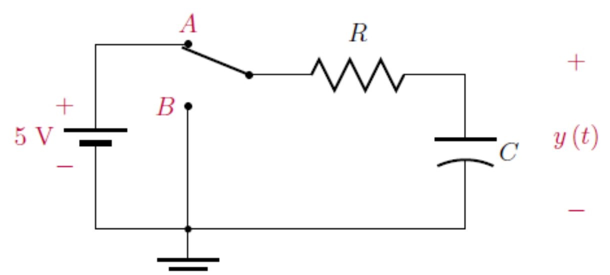
The constant c must be determined based on the desired initial value of $y(t)$ at $t = t_0$.

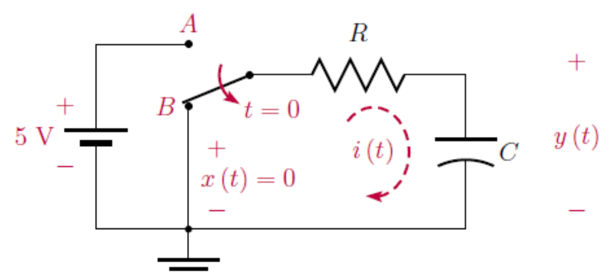
2.5 Solving Differential Equations

Example 2.11

Natural response of the simple RC circuit

Consider the RC circuit shown. Element values are $R = 1\ \Omega$ and $C = 1/4$ F. Input terminals of the circuit are connected to a battery that supplies the circuit with an input voltage of 5 V up to the time instant $t = 0$. The switch is moved from position A to position B at $t = 0$ ensuring that $x(t) = 0$ for $t \geq 0$. Find the output signal as a function of time.





Solution:

$$RC\,\frac{dy(t)}{dt} + y(t) = x(t)$$

$$\frac{dy(t)}{dt} + \frac{1}{RC}\,y(t) = \frac{1}{RC}\,x(t)$$

Homogeneous differential equation:

$$\frac{dy(t)}{dt} + 4\,y(t) = 0$$

2.5 Solving Differential Equations

Solution: Since $x(t) = 0$ for $t > 0$, the output signal is $y(t) = y_h(t)$, and we are trying to find the homogeneous solution. Substituting the specified parameter values, the homogeneous differential equation is found as

$$\frac{dy(t)}{dt} + 4\,y(t) = 0$$

We need $(s + 4) = 0$, and the corresponding homogeneous solution is in the form

$$y_h(t) = c\,e^{-st} = c\,e^{-4t}$$

for $t \geq 0$. The initial condition $y_h(0) = 5$ must be satisfied. Substituting $t = 0$ into the homogeneous solution, we get

$$y_h(0) = c\,e^{-4(0)} = c = 5$$

Using the value found for the constant c, the natural response of the circuit is

$$y_h(t) = 5\,e^{-4t}\,, \quad \text{for} \quad t \geq 0$$

The natural response can be expressed in a more compact form through the use of the unit-step function:

$$y_h(t) = 5\,e^{-4t}\,u(t) \tag{2.78}$$

2.5 Solving Differential Equations

Example 2.11 (continued)

Homogeneous solution is of the form:

$$y_h(t) = c\,e^{-st} = c\,e^{-4t}\,, \qquad \text{for } t \geq 0$$

Satisfy initial value:

$$y_h(0) = c\,e^{-4(0)} = c = 5$$

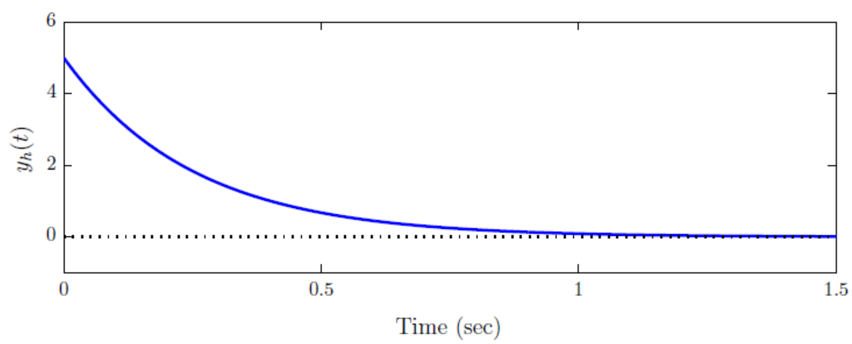
Natural response:

$$y_h(t) = 5\,e^{-4t}\,, \quad \text{for } \ t \geq 0$$

In compact form:

$$y_h(t) = 5\,e^{-4t}\,u(t)$$

$$y_h(t) = 5\,e^{-4t}\,u(t)$$



2.5 Solving Differential Equations

Finding the forced response of a continuous-time system

Choosing a particular solution for various input signals

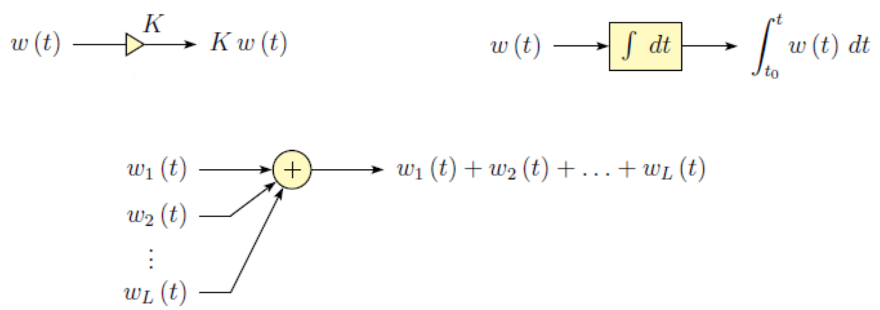
Input signal	Particular solution
K (constant)	k_1
$K\,e^{at}$	$k_1\,e^{at}$
$K\ \cos(at)$	$k_1\ \cos(at) + k_2\ \sin(at)$
$K\ \sin(at)$	$k_1\ \cos(at) + k_2\ \sin(at)$
$K\,t^n$	$k_n t^n + k_{n-1} t^{n-1} + \ldots + k_1 t + k_0$

2.6 Block Diagram Representation of Continuous-Time Systems

Block diagram representation of continuous-time systems

Block diagrams for continuous-time systems are constructed using three types of components:

- Constant-gain amplifiers
- Signal adders
- Integrators



2.6 Block Diagram Representation of Continuous-Time Systems

Block diagram representation of continuous-time systems (continued)

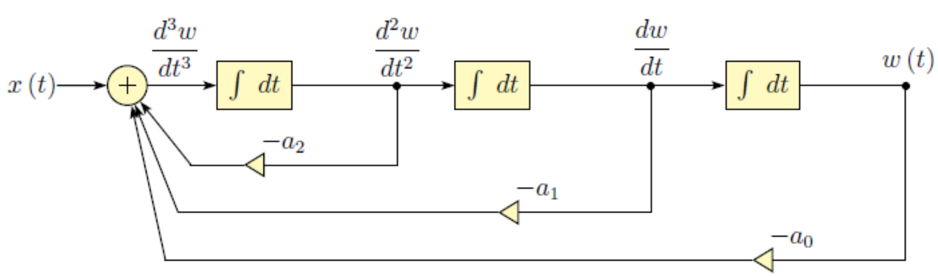
A third-order differential equation:

$$\frac{d^3y}{dt^3} + a_2\,\frac{d^2y}{dt^2} + a_1\,\frac{dy}{dt} + a_0\,y = b_2\,\frac{d^2x}{dt^2} + b_1\,\frac{dx}{dt} + b_0\,x$$

Use an intermediate variable $w(t)$ in place of $y(t)$ in the left side of the differential equation, and set the result equal to $x(t)$:

$$\frac{d^3w}{dt^3} + a_2\,\frac{d^2w}{dt^2} + a_1\,\frac{dw}{dt} + a_0\,w = x$$

$$\frac{d^3w}{dt^3} = x - a_2\,\frac{d^2w}{dt^2} - a_1\,\frac{dw}{dt} - a_0\,w$$

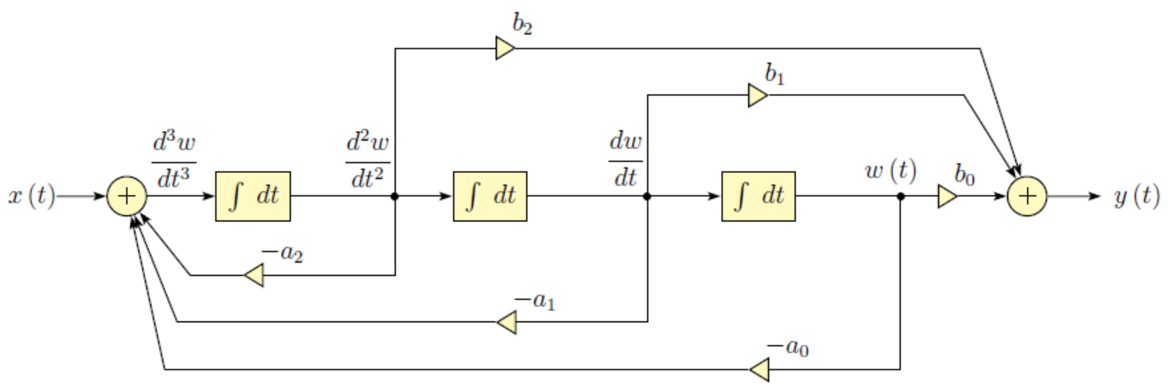


2.6 Block Diagram Representation of Continuous-Time Systems

Block diagram representation of continuous-time systems (continued)

Express the signal $y(t)$ in terms of the intermediate variable $w(t)$:

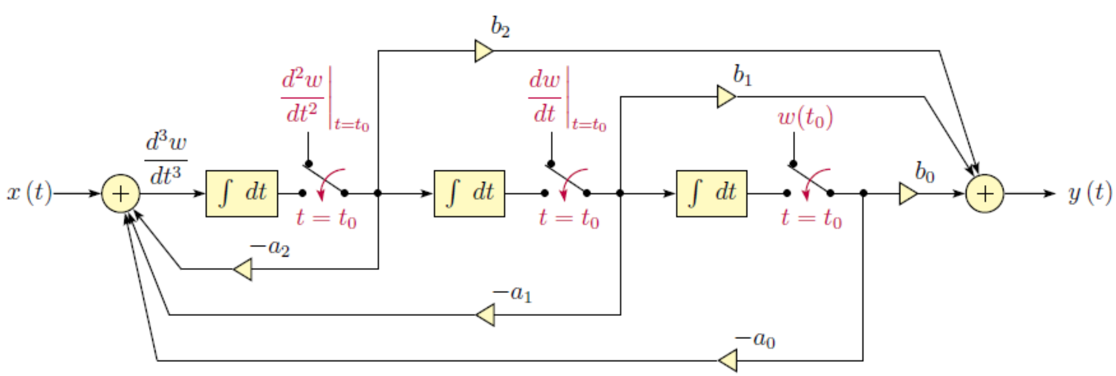
$$y = b_2 \frac{d^2 w}{dt^2} + b_1 \frac{dw}{dt} + b_0 \, w$$

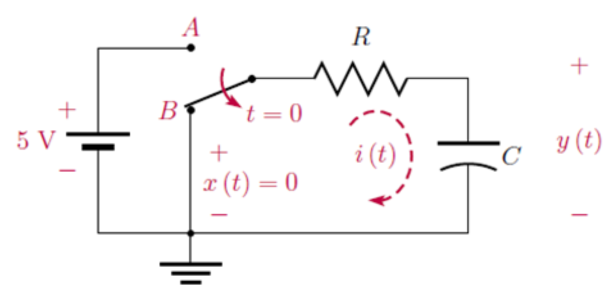


2.6 Block Diagram Representation of Continuous-Time Systems

Block diagram representation of continuous-time systems (continued)

Imposing initial conditions:





2.6 Block Diagram Representation of Continuous-Time Systems

The proof for Eqn. (2.122) will be given starting with the right side of Eqn. (2.119) and expressing the terms in it through the use of Eqn. (2.120). We have the following relationships:

$$b_0\, x = b_0 \left(\frac{d^3 w}{dt^3} + a_2 \frac{d^2 w}{dt^2} + a_1 \frac{dw}{dt} + a_0\, w \right) \tag{2.123}$$

$$b_1 \frac{dx}{dt} = b_1 \left(\frac{d^4 w}{dt^4} + a_2 \frac{d^3 w}{dt^3} + a_1 \frac{d^2 w}{dt^2} + a_0 \frac{dw}{dt} \right) \tag{2.124}$$

$$b_2 \frac{d^2 x}{dt^2} = b_2 \left(\frac{d^5 w}{dt^5} + a_2 \frac{d^4 w}{dt^4} + a_1 \frac{d^3 w}{dt^3} + a_0 \frac{d^2 w}{dt^2} \right) \tag{2.125}$$

Adding Eqns. (2.123) through (2.125) we get

$$\begin{aligned} b_0\, x + b_1 \frac{dx}{dt} + b_2 \frac{d^2 x}{dt^2} = & \left(b_0 \frac{d^3 w}{dt^3} + b_1 \frac{d^4 w}{dt^4} + b_2 \frac{d^5 w}{dt^5} \right) \\ & + a_2 \left(b_0 \frac{d^2 w}{dt^2} + b_1 \frac{d^3 w}{dt^3} + b_2 \frac{d^4 w}{dt^4} \right) \\ & + a_1 \left(b_0 \frac{dw}{dt} + b_1 \frac{d^2 w}{dt^2} + b_2 \frac{d^3 w}{dt^3} \right) \\ & + a_0 \left(b_0\, w + b_1 \frac{dw}{dt} + b_2 \frac{d^2 w}{dt^2} \right) \\ = & \frac{d^3 y}{dt^3} + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0\, y \end{aligned} \tag{2.126}$$

proving that Eqns. (2.120) and (2.122) are indeed equivalent to Eqn. (2.119). Thus, the output $y\,(t)$ can be obtained through the use of Eqn. (2.122).

2.7 Impulse Response and Convolution

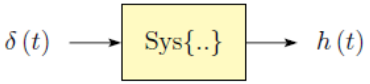
- We have explored the use of differential equations for describing the time-domain behavior of continuous-time systems and have concluded that a ***CTLTI system can be completely described by means of a constant-coefficient ordinary differential equation***.

- ***An alternative description*** of a CTLTI system can be given in terms of its impulse response h (t) which is simply the *forced response of the system under consideration when the input signal is a unit impulse*.

2.7 Impulse Response and Convolution

Impulse response

$$h(t) = \text{Sys}\{\delta(t)\}$$



For a CTLTI system: The impulse response also constitutes a complete description of the system.

Finding the impulse response of a CTLTI system from the differential equation

1. Use a unit-step function for the input signal, and compute the forced response of the system, i.e., the *unit-step response*.
2. Differentiate the unit-step response of the system to obtain the impulse response, i.e.,

$$h(t) = \frac{dy(t)}{dt}$$

$$\text{Sys}\{\delta(t)\} = \text{Sys}\left\{\frac{du(t)}{dt}\right\} = \frac{d}{dt}\left[\text{Sys}\{u(t)\}\right]$$

This idea relies on the fact that differentiation is a *linear operator*.

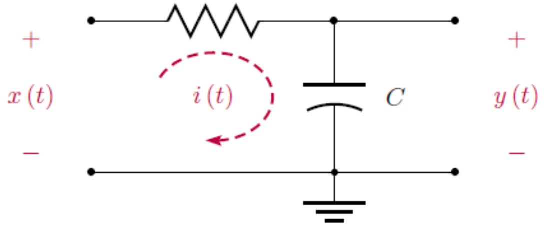
2.7 Impulse Response and Convolution

Example 2.18

Impulse response of the simple RC circuit

Determine the impulse response of the first-order RC circuit shown. Assume the system is initially relaxed, that is, there is no initial energy stored in the system. (Recall that this is a necessary condition for the system to be CTLTI.)

$$RC\,\frac{dy(t)}{dt} + y(t) = x(t)$$



Solution: Differential equation is

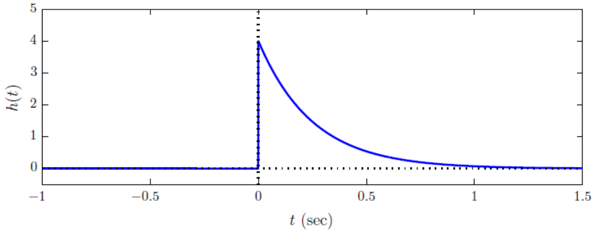
$$\frac{dy(t)}{dt} + 4\,y(t) = 4\,x(t) \qquad \frac{dy(t)}{dt} + \frac{1}{RC}\,y(t) = \frac{1}{RC}\,x(t)$$

Using the first-order solution method:

$$h(t) = \int_0^t e^{-4(t-\tau)}\,4\,\delta(\tau)\,d\tau$$

Using the sifting property of the unit-impulse function:

$$h(t) = 4\,e^{-4t}\,u(t)$$



2.7 Impulse Response and Convolution

Solution: We will solve this problem using two different methods. The differential equation for the circuit was given by Eqn. (2.17). Since we have a first-order differential equation, we can use the solution method that led to Eqn. (2.55) in Section 2.5.1. Letting $\alpha = 1/RC$, $t_0 = 0$ and $y(0) = 0$ we have

$$y(t) = \int_0^t e^{-(t-\tau)/RC} \, \frac{1}{RC} \, x(\tau) \, d\tau \tag{2.138}$$

If the input signal is chosen to be a unit-impulse signal, then the output signal becomes the impulse response of the system. Setting $x(t) = \delta(t)$ leads to

$$h(t) = \int_0^t e^{-(t-\tau)/RC} \, \frac{1}{RC} \, \delta(\tau) \, d\tau \tag{2.139}$$

Using the sifting property of the unit-impulse function, we obtain

$$h(t) = \frac{1}{RC} \, e^{-t/RC} \, u(t)$$

Now we will obtain the same result using the more general method developed in Section 2.7.1. Recall that the unit-step response of the system was found in Example 2.8 Eqn. (2.59) as

$$y(t) = \left(1 - e^{-t/RC}\right) u(t)$$

Differentiating Eqn. (2.140) with respect to time we obtain the impulse response:

$$h(t) = \frac{dy(t)}{dt} = \frac{d}{dt}\left[\left(1 - e^{-t/RC}\right) u(t)\right] = \frac{1}{RC} \, e^{-t/RC} \, u(t)$$

which is the same result obtained in Eqn. (2.140). With the substitution of the specified element values, the impulse response becomes

$$h(t) = 4 \, e^{-4t} \, u(t) \tag{2.140}$$

2.7 Impulse Response and Convolution

Convolution operation for CTLTI systems

The output signal $y(t)$ of a CTLTI system is equal to the convolution of its impulse response $h(t)$ with the input signal $x(t)$.

Continuous-time convolution

$$
\begin{aligned}
y(t) &= x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda \\
&= h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda
\end{aligned}
$$

2.7 Impulse Response and Convolution

Convolution operation for CTLTI systems (continued)

Steps involved in computing the convolution of two signals

To compute the convolution of $x(t)$ and $h(t)$ at a specific time-instant t:

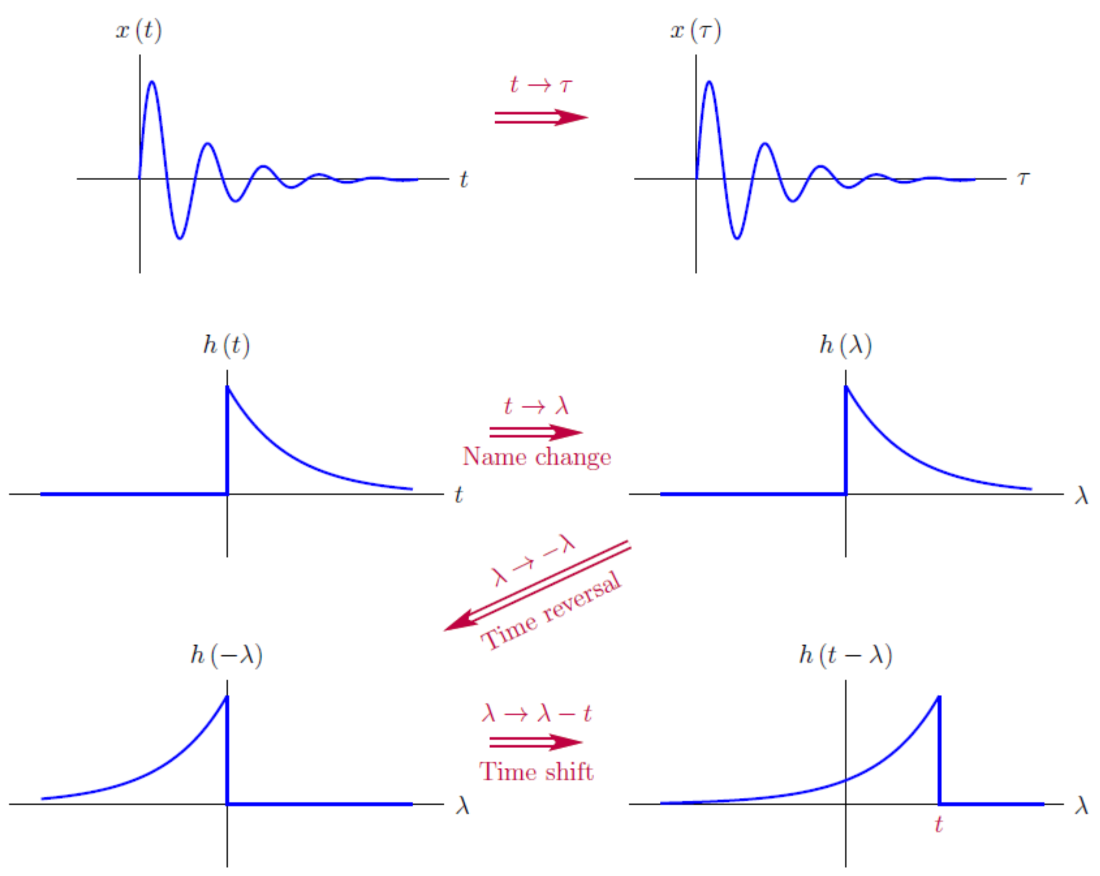
1. Sketch the signal $x(\lambda)$ as a function of the independent variable λ. This corresponds to a simple name change on the independent variable, and the graph of the signal $x(\lambda)$ appears identical to the graph of the signal $x(t)$.
2. For one specific value of t, sketch the signal $h(t-\lambda)$ as a function of the independent variable λ. This task can be broken down into two steps as follows:
   - 2a. Sketch $h(-\lambda)$ as a function of λ. This step amounts to time-reversal of $h(\lambda)$.
   - 2b. In $h(\lambda)$ substitute $\lambda \rightarrow \lambda - t$. This step yields

     $$h(-\lambda)\Big|_{\lambda\rightarrow\lambda-t} = h(t-\lambda)$$

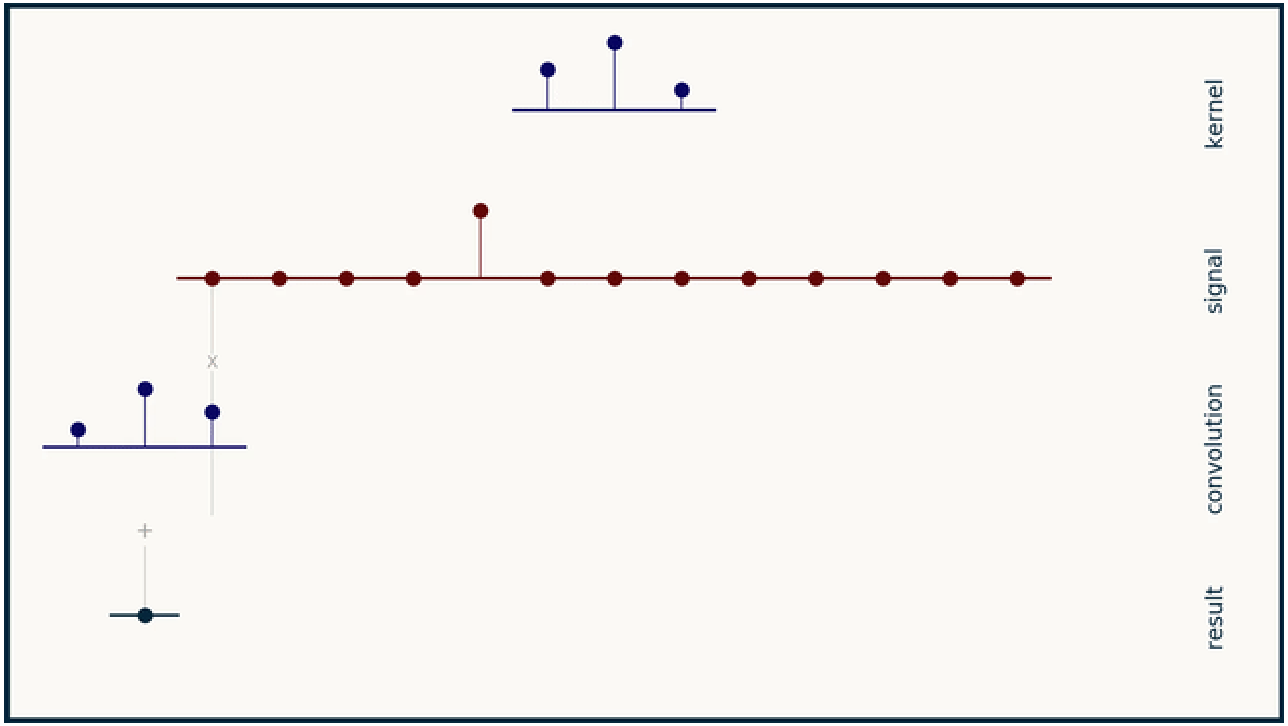
     and amounts to time-shifting $h(-\lambda)$ by t.
3. Multiply the two signals in 1 and 2 to obtain $f(\lambda) = x(\lambda)\, h(t-\lambda)$.
4. Compute the area under the product $f(\lambda) = x(\lambda)\, h(t-\lambda)$ by integrating it over the independent variable λ. The result is the value of the output signal at the specific time instant t.
5. Repeat steps 1 through 4 for all values of t that are of interest.

2.7 Impulse Response and Convolution

Convolution operation for CTLTI systems (continued)



2.7 Impulse Response and Convolution



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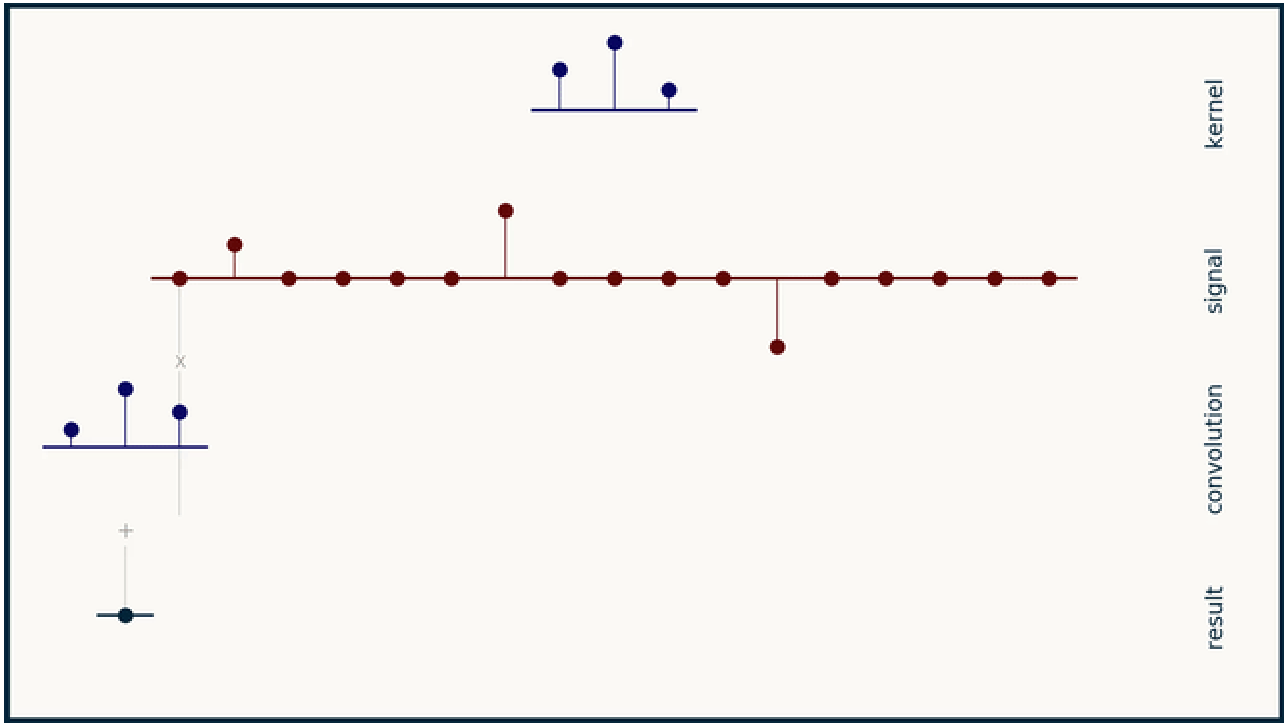
2.7 Impulse Response and Convolution



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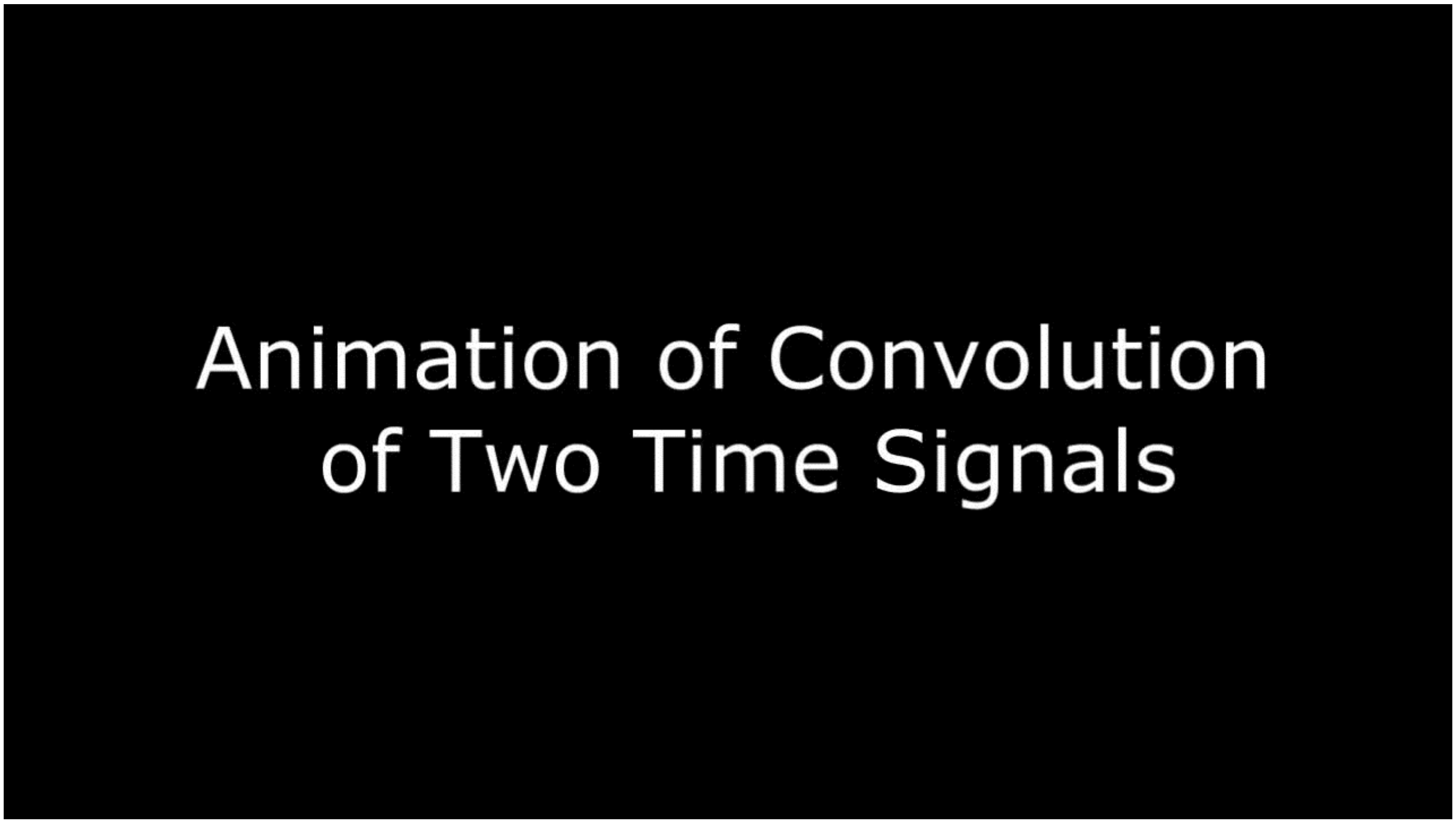
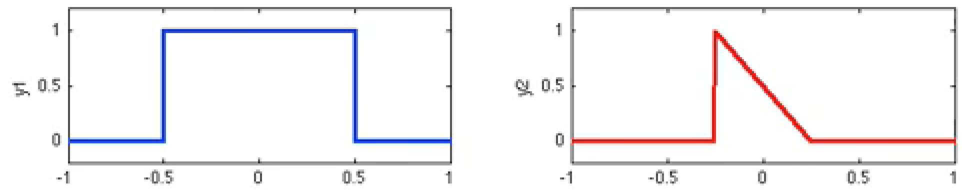
2.7 Impulse Response and Convolution



2.7 Impulse Response and Convolution

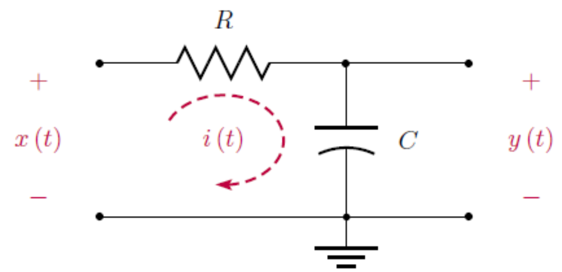


2.7 Impulse Response and Convolution

Example 2.20

Unit-step response of RC circuit revisited

Compute the unit-step response of the simple RC circuit using the convolution operation.



Solution:

Impulse response of the RC circuit is

$$h(t) = \frac{1}{RC}\, e^{-t/RC}\, u(t)$$

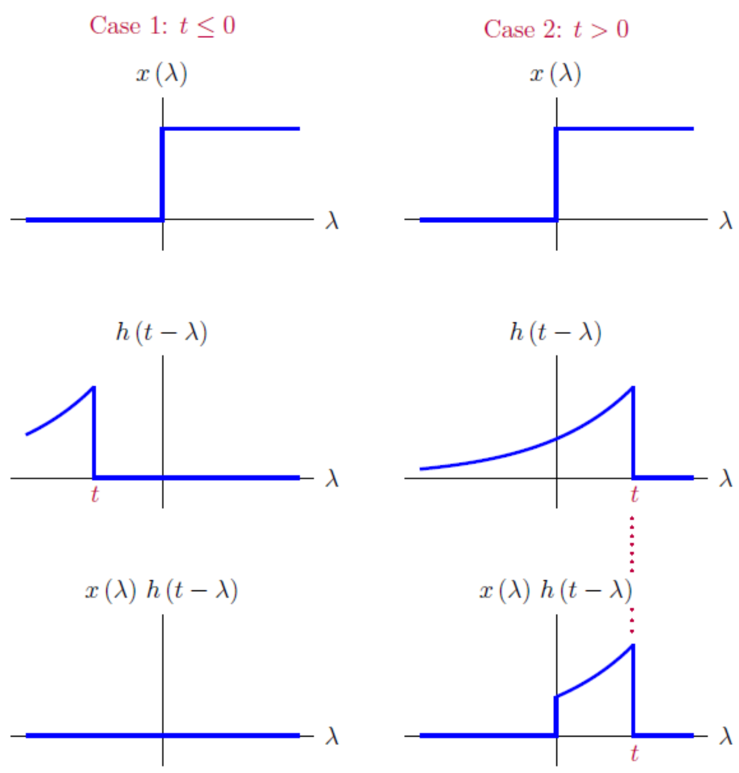
Output of the system in response to input $x(t)$:

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

Functions needed: $x(\lambda)$ and $h(t-\lambda)$.

2.7 Impulse Response and Convolution

Example 2.20 (continued)



2.7 Impulse Response and Convolution

Example 2.20 (continued)

Case 1: $t \le 0$

Functions $x(\lambda)$ and $h(t-\lambda)$ do not overlap anywhere. Therefore

$$y(t) = 0, \quad \text{for } t \le 0$$

Case 2: $t > 0$

Functions $x(\lambda)$ and $h(t-\lambda)$ overlap for values of λ in the interval $(0, t)$. In this interval $x(\lambda) = 1$ and $h(t-\lambda) = \frac{1}{RC}\, e^{-(t-\lambda)/RC}$. Therefore

$$y(t) = \int_0^t \frac{1}{RC}\, e^{-(t-\lambda)/RC}\, d\lambda = 1 - e^{-t/RC}, \quad \text{for} \quad t > 0$$

Combine the two cases through the use of a unit-step function:

$$y(t) = \left(1 - e^{-t/RC}\right) u(t)$$

2.8 Causality in Continuous-Time Systems

Causality in continuous-time systems

Causal system

A system is said to be causal if the current value of the output signal depends only on current and past values of the input signal, but not on its future values.

CTLTI system:

$$y\,(t) = h\,(t) * x\,(t) = \int_{-\infty}^{\infty} h\,(\lambda)\; x\,(t-\lambda)\; d\lambda$$

For $\lambda < 0$, the term $x\,(t-\lambda)$ refers to future values of the input signal.

Causality in CTLTI systems

For a CTLTI system to be causal, the impulse response of the system must be equal to zero for all negative values of its argument.

$$h\,(t) = 0 \quad \text{for all} \quad t < 0$$

2.8 Stability in Continuous-Time Systems

Stability in continuous-time systems

Stable system

A system is said to be *stable* in the *bounded-input bounded-output (BIBO)* sense if any bounded input signal is guaranteed to produce a bounded output signal.

$$\left| x(t) \right| < B_x < \infty \quad \text{implies that} \quad \left| y(t) \right| < B_y < \infty$$

CTLTI system:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

Stability in CTLTI systems

For a CTLTI system to be stable, its impulse response must be *absolute integrable*.

$$\int_{-\infty}^{\infty} \left| h(\lambda) \right| d\lambda < \infty$$

Conclusion